Hat Puzzle

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1] Problem Statement

One evening at the campfire, Dudley the Detective pulled out a bag with a mysterious bundle inside. “In this bag,” he told his friends, “are five hats. Two are yellow and three are blue. Now I want three volunteers for a serious detective puzzle.”
“Each of you must guess the color of the hat on your own head. You may not look in the pond for your reflection, but you must use your deductive thinking skills. You may, of course, look at the hats on the other two heads.”
Max looked at both of his friends and the hats on their heads. “But I can’t tell what color hat I’m wearing,” he said in a puzzled voice.
Jordan agreed as he looked wisely around that he couldn’t tell the color of his hat either.
Sybil smiled broadly and instantly blurted, “Then I DO know the color of my hat!”
What color hat was Sybil wearing?

2] Preface

From the described scenario we know that a person who observes two yellow hats can deduce that they are wearing a blue hat because there are no more remaining yellow hats.

Theorem 2.1. A person who sees two yellow hats can deduce that they are wearing a blue hat.

Max and Jordan are unable to ascertain the color of their own hat. For simplicity, let us assume that they are perfectly rational beings and they have the mental faculties to grasp all the information they need in this situation. There are many logical statements that derive from this assumption; for the simplicity of this informal proof we will only select two logical statements relative to our proof. Our first assumption will be that if Max saw Jordan and Sybil wearing yellow hats, then Max would know that his hat is blue. Our second assumption, which may seem a bit strange, is constructed as follows: If Jordan sees Sybil wearing a yellow hat and he knows that Sybil wearing a yellow hat implies that he is wearing a blue hat, then Jordan would know that he is wearing a blue hat. Our last assumption, which may seem trivial, is that Max sees Sybil wearing a blue if and only if Jordan sees Sybil wearing a blue hat.
With the described assumptions, we can construct an informal proof.

3] Informal Proof

Let us define the literals j_m, s_m, and s_j to respectively denote the propositional statements "Max sees Jordan wearing a blue hat," "Max sees Sybil wearing a blue hat," and "Jordan sees Sybil wearing a blue hat." Let us also define the literals k_m and k_j to respectively denote the propositional statements "Max knows the color of his hat," and "Jordan knows the color of his hat."

1.	$\neg k_m$	*Premise*
2.	$\neg k_j$	*Premise*
3.	$s_m \leftrightarrow s_j$	*Premise*
4.	$(\neg j_m \wedge \neg s_m) \rightarrow k_m$	*Premise*
5.	$(\neg s_j \wedge (\neg s_j \rightarrow j_m)) \rightarrow k_j$	*Premise*
6.	$\neg(\neg j_m \wedge \neg s_m)$	(1), (4), *Modus Tollens*
7.	$\neg\neg(\neg\neg j_m \vee \neg\neg s_m)$	(6), *Modus Ponens*, *Conjunction Negation* (*DeMorgan*)
8.	$j_m \vee s_m$	(7), *Modus Ponens*, *Double Negation*
9.	$(\neg s_j \wedge (\neg\neg s_j \vee j_m)) \rightarrow k_j$	(5), *Modus Ponens*, *Implication Simplification*
10.	$(\neg s_j \wedge (s_j \vee j_m)) \rightarrow k_j$	(9), *Modus Ponens*, *Double Negation*
11.	$\neg(\neg s_j \wedge (s_j \vee j_m))$	(2), (10), *Modus Tollens*
12.	$\neg\neg(\neg\neg s_j \vee \neg(s_j \vee j_m))$	(11), *Modus Ponens*, *Conjunction Negation* (*DeMorgan*)
13.	$s_j \vee \neg(s_j \vee j_m)$	(12), *Modus Ponens*, *Double Negation*
14.	$j_m \vee (s_m \vee s_j)$	(8), *Modus Ponens*, *Disjunction Introduction*
15.	$j_m \vee (s_m \wedge s_j)$	(3), (14), *Modus Ponens*, *Biconditional Elimination*
16.	$(j_m \vee s_m) \wedge (j_m \vee s_j)$	(15), *Modus Ponens*, *Distribution*
17.	$(j_m \vee s_j)$	(16), *Modus Ponens*, *Conjunction Elimination*
18.	s_j	(13), (17), *Modus Ponens*, *Disjunctive Syllogism*

Thus, given the described scenario, we are able to conclude that s_j is true. In other words, Jordan saw Sybil wearing a blue hat.